

Stalling chaos control accelerates convergence

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Abstract. Since chaos control has found its way into many applications, the development of fast, easy-to-implement and universally applicable chaos control methods is of crucial importance. Predictive feedback control has been widely applied but suffers from a speed limit imposed by highly unstable periodic orbits. We show that this limit can be overcome by stalling the control, thereby taking advantage of the stable directions of the uncontrolled chaotic map. This analytical finding is confirmed by numerical simulations, giving a chaos-control method that is capable of successfully stabilizing periodic orbits of high period.

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1. Introduction

Chaotic attractors of dynamical systems typically contain infinitely many periodic orbits [1]. The goal of chaos control is to render these orbits stable through suitable perturbations of the system. Since the seminal work by Ott, Grebogi and Yorke (OGY) [2], such chaos control has given rise to many applications [3] including chaotic lasers, stabilizing cardiac rhythms, biological [4] and also artificial neural dynamics and autonomous robot control [5].

The original OGY control method employs arbitrarily small perturbations of an accessible control parameter of the system to move the trajectory onto the stable manifold of a known periodic orbit. A different approach similar to time-delayed feedback control for time-continuous systems [6] is given by predictive feedback control (PFC) [7–9] that requires less prior knowledge about the system. Here, predictions of the future state of the dynamics are fed back to the system in the form of a control signal to stabilize unstable periodic orbits. In contrast to the OGY method, it does not require either *a priori* knowledge about or online sampling to determine the periodic orbits and their stability properties as it acts as a global stability transformation [10, 11]. Moreover, it is non-invasive, i.e. the control strength vanishes upon convergence. Hence, PFC provides a universally applicable, easy-to-implement method which requires little prior knowledge about the system and has been successfully applied.

In any real-world application, convergence speed is important. For example, if chaos control is used to stabilize cardiac rhythms [12], convergence speed is crucial. This aspect of a successful implementation of chaos control is mostly overlooked in the literature as the main focus is on maximizing the number of periodic orbits that can be stabilized. The convergence speed of PFC depends on the stability properties of the periodic orbits and convergence speed decreases for increasingly unstable periodic orbits. Hence, as periodic orbits of chaotic attractors of large period are generally highly unstable [13], PFC becomes increasingly unfeasible in applications. Any adaptation method within the PFC framework [9] operates within this speed limit.

In this paper we show that this speed limit can be overcome by ‘stalling’ PFC, i.e. repeatedly making use of the uncontrolled, chaotic dynamics of the system. This way, we obtain a period-independent scaling of asymptotic convergence speed for periodic orbits when classified by their period. This increase in speed is achieved while maintaining the method’s nature of being a simple easy-to-implement one-parameter control scheme. Even though related ideas were put forth [14, 15], for example in implementation-related efforts to stabilize periodic orbits, the aspect of convergence speed was not considered. It is our general formulation through the introduction of an arbitrary stalling parameter presented here which admits overall fast

stabilization of unstable periodic orbits. Moreover, the increase in optimal convergence speed is not limited to the linearized dynamics. It persists if the initial conditions are sampled randomly on the chaotic attractor as we illustrate by numerical simulations for the Hénon map.

2. Speed limit of predictive feedback control

Suppose that $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a differentiable map. Furthermore, assume that the iteration of f gives rise to a chaotic attractor with a dense set of unstable periodic orbits. Fix a natural number $p \in \mathbb{N}$ and let $f_p = f^{\circ p}$ denote the p -fold iterate of f . The set of periodic orbits of (minimal) period p of f is given by $\text{Fix}(f, p) = \{x^* \in \mathbb{R}^N \mid f_p(x^*) = x^*, f_q(x^*) \neq x^* \text{ for } q < p\}$. The elements of $\text{Fix}(f, 1)$ are called fixed points of f . Each x^* is a fixed point of the iteration defined through the evolution equation

$$x_{k+1} = f_p(x_k) \quad (1)$$

for $k \in \mathbb{N}$. In other words, to study periodic orbits of the iteration of f we study the fixed points of f_p .

By employing the transformation $S(\mu) : f_p \mapsto \text{id} + \mu(f_p - \text{id}) = g_{\mu,p}$ (cf [11]) where id is the identity map on \mathbb{R}^N and $\mu \in [-1, 1]$, we obtain the predictive feedback chaos control method given by the iteration

$$x_{k+1} = g_{\mu,p}(x_k) = f_p(x_k) + \eta(x_k - f_p(x_k))$$

with $\eta = 1 - \mu$. Any periodic orbit of period p of f is a fixed point of $g_{\mu,p}$ (and vice versa if $\eta \neq 1$). Let $df_p|_x$ denote the total derivative of f_p at $x \in \mathbb{R}^N$. Let $\text{Fix}^*(f, p) \subset \text{Fix}(f, p)$ denote the set of periodic orbits where both $df_p|_{x^*}$ and $(df_p|_{x^*} - \text{id})$ are non-singular and diagonalizable (over \mathbb{C}). There is a maximal subset $\text{Fix}_g^*(f, p) \subset \text{Fix}^*(f, p)$ with the property that for every $x^* \in \text{Fix}_g^*(f, p)$ there exists a $\mu \in [-1, 1]$ such that x^* is a stable fixed point of $g_{\mu,p}$. The set $\text{Fix}_g^*(f, p)$ is called the set of PFC-stabilizable periodic orbits. It is usually not empty and its elements x^* are identified by the local stability properties of $df_p|_{x^*}$. For a two-dimensional system, these are the saddles with negative eigenvalue corresponding to the unstable direction [11, 16].

The local stability properties of $g_{\mu,p}$ are readily computed. Suppose that $x^* \in \text{Fix}_g^*(f, p)$ and $\lambda_j \in \mathbb{C}$, $j \in \{1, \dots, N\}$, are the eigenvalues of $df_p|_{x^*}$. Since the Jacobian of $g_{\mu,p}$ at x^* is given by $dg_{\mu,p}|_{x^*} = \text{id} + \mu(df_p|_{x^*} - \text{id})$, its eigenvalues evaluate to

$$\kappa_j(\mu) = 1 + \mu(\lambda_j - 1). \quad (2)$$

This relationship between the local stability properties of the original and transformed systems implies a speed limit for the asymptotic convergence speed of PFC. Consider a saddle $x^* \in \text{Fix}_g^*(f, p)$ of a two-dimensional map as described above with $\lambda_1 \in (-1, 1)$ and $\lambda_2 < -1$. To stabilize x^* choose $\mu_0 > 0$ such that $\kappa_2(\mu_0) = 1 + \mu_0(\lambda_2 - 1) > -1$ and hence $0 < \mu_0 < 2(1 - \lambda_2)^{-1}$. This implies that if x^* is highly unstable with $|\lambda_2| \gg 1$ the control parameter μ_0 has to be chosen small enough for x^* to become stable for $g_{\mu_0,p}$. Recall that the spectral radius of the linearization at a fixed point determines asymptotic convergence speed and convergence becomes increasingly slow as it approaches one. For $\mu \rightarrow 0$ we have $\kappa_1(\mu) \rightarrow 1$ and thus, the spectral radius $\varrho(dg_{\mu,p}|_{x^*}) = \max_{j \in \{1, \dots, N\}} |\kappa_j(\mu)| \rightarrow 1$ converges to one as $|\lambda_2| \rightarrow \infty$. In other words, increasing instability, which is exhibited in particular for periodic orbits of large periods [13], implies lower asymptotic convergence speed due to the slow convergence along the

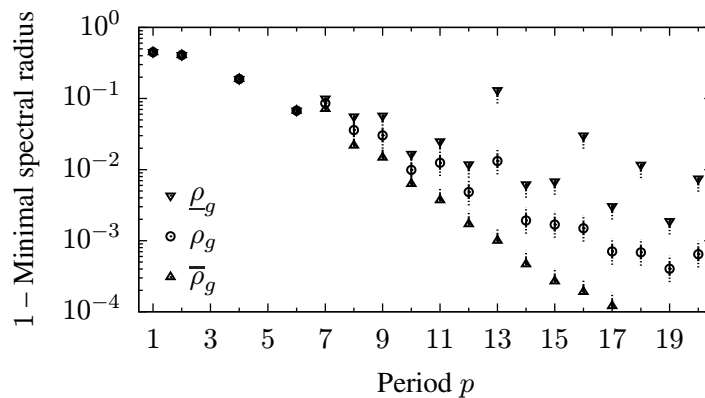


Figure 1. Slowdown of predictive feedback control (PFC). The optimal asymptotic convergence speed for the (PFC-stabilizable) periodic orbits of the Hénon map (3) stabilized by PFC decreases exponentially on average for increasing period. Note that, there are no PFC-stabilizable orbits of periods three and five.

originally stable direction. The analogous argument holds for PFC-stabilizable periodic orbits in higher dimensions.

To illustrate this phenomenon, we calculated this slowdown explicitly for the Hénon map $H : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with parameters $a, b \in \mathbb{R}$ given by

$$H(x_1, x_2) = (x_2 + 1 - ax_1^2, bx_1), \quad (3)$$

which has a chaotic attractor for $a = 1.4$ and $b = 0.3$ [17]. Let $\varrho_{\min}^g(x^*) = \inf_{\mu} \varrho(\mathrm{d}g_{\mu,p}|_{x^*})$ denote the spectral radius of the linearization at a periodic orbit x^* for the optimal parameter value and $\langle \cdot \rangle_X$ the population mean over a finite set X . The average asymptotic convergence speed, which can be assessed by evaluating

$$\rho_g(p) = 1 - \langle \varrho_{\min}^g(x^*) \rangle_{\mathrm{Fix}_g^*(f,p)}, \quad (4)$$

clearly decreases (exponentially) with increasing period (figure 1) due to the increasing instability of the unstable periodic orbits of higher period. Similar scaling is observed for best and worst asymptotic convergence speed across periods by evaluating the lower bound $\underline{\rho}_g(p)$ and upper bound $\bar{\rho}_g(p)$ of $1 - \varrho_{\min}^g(x^*)$ for $x^* \in \mathrm{Fix}_g^*(f,p)$. Thus, PFC becomes less feasible as the period of the orbits increases and convergence is dominated by slow convergence along the originally stable direction.

3. Can this speed limit be broken?

It is the cause of the speed limit that motivates ‘stalling’ PFC. Note that the eigenvectors of $\mathrm{d}f_p|_x$ and $\mathrm{d}g_{\mu,p}|_x$ are the same for any $x \in \mathbb{R}^N$. Convergence towards a stable fixed point takes place along the leading direction, i.e. the eigenvector corresponding to the eigenvalue of largest absolute value. This means that the originally stable directions of f_p at a periodic orbit x^* are the leading directions of the transformed system. Hence, for an initial condition x_0 close to a periodic orbit, $g_{\mu,p}$ brings the trajectory close to the stable manifold. Applying the map f_p will take the trajectory along the stable manifold closer to the periodic orbit (while diverging from

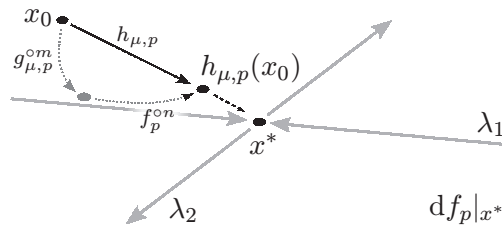


Figure 2. Increase of asymptotic convergence speed by stalling PFC. The PFC-transformed map $g_{\mu,p}$ takes a trajectory closer to the stable manifold of f at a periodic orbit x^* . Iterating (5) takes advantage of this by fast convergence along the stable manifold by f_p .

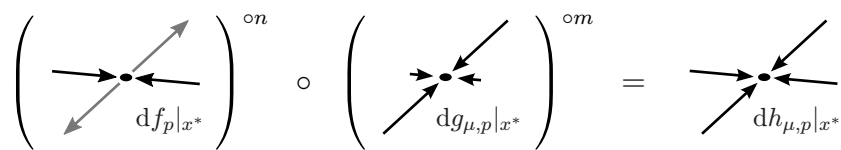


Figure 3. Stalling PFC leads to fast convergence from all directions to a periodic orbit x^* for small μ . In this cartoon, for two dimensions the direction of the arrows indicates stability and the length the absolute value of the corresponding eigenvalue.

the stable manifold), cf figure 2. Adding evaluations of the uncontrolled map f_p (i.e. ‘stalling’ control) should therefore yield increased convergence speed as this iteration makes use of the fast convergence along the stable manifold, resulting in local stability properties as illustrated in figure 3. In some sense, this iteration is similar to OGY control, the goal of which is to place the trajectory on the stable manifold. Here, however, no explicit knowledge of the periodic orbit and its local stability is needed because we exploit of the global nature of the stability transformation $S(\mu)$.

Stalled PFC (SPFC) is defined as follows. For a map ψ define $\psi^{\circ 0} = \text{id}$. Set

$$h_{\mu,p}^{(m,n)} = h_{\mu,p} := (f_p)^{\circ n} \circ (g_{\mu,p})^{\circ m} \quad (5)$$

with parameters $m, n \in \mathbb{N} \cup \{0\}$. We omit the parameters m, n unless the choice is important. Let $x^* \in \text{Fix}^*(f, p)$ be a periodic orbit of period p . With local stability of f_p at x^* given by λ_j and of $g_{\mu,p}$ at x^* given by (2), the eigenvalues of $dh_{\mu,p}|_{x^*}$ evaluate to

$$\Lambda_j = \lambda_j^n \kappa_j(\mu)^m = \lambda_j^n (1 + \mu(\lambda_j - 1))^m \quad (6)$$

for all $j \in \{1, \dots, N\}$. To be able to compare convergence speed, the absolute value of the eigenvalues needs to be rescaled since $h_{\mu,p}$ contains n evaluations of f_p and m evaluations of $g_{\mu,p}$. With a general stalling parameter $\alpha = n(n+m)^{-1}$ we obtain the values

$$l_j(\alpha, \mu) = |\lambda_j|^\alpha |1 + \mu(\lambda_j - 1)|^{1-\alpha} \quad (7)$$

that determine local stability rescaled to one evaluation of f_p . Hence, rescaled asymptotic convergence speed is given by the stability function

$$\mathcal{Q}_{x^*}(\alpha, \mu) = \max_{j \in \{1, \dots, N\}} l_j(\alpha, \mu). \quad (8)$$

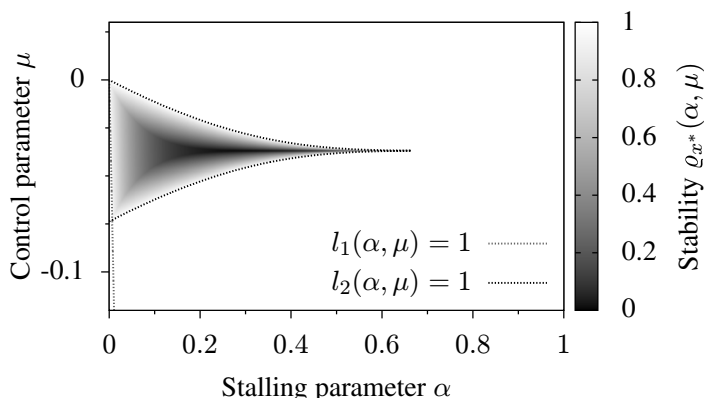


Figure 4. SPFC parameter space reveals regions of fast convergence. The shading depicts the stability function (8) for a periodic orbit of period $p = 6$ of the Hénon map (3) with local stability given by $(\lambda_1, \lambda_2) = (2.592 \times 10^{-5}, 28.125)$. Note that this periodic orbit is SPFC-stabilizable but not PFC-stabilizable (i.e. $\text{Fix}_g^*(f, p) \neq \text{Fix}_h^*(f, p)$ for the Hénon map) since the original PFC method is recovered for $\alpha = 0$ and we have $\varrho_{x^*}(0, \mu) \geq 1$.

A periodic orbit x^* is called SPFC-stabilizable if there are parameters $\alpha_0 \in \mathbb{Q} \cap [0, 1]$ (with \mathbb{Q} denoting the rational numbers) and $\mu_0 \in [-1, 1]$ such that $\varrho_{x^*}(\alpha_0, \mu_0) < 1$. The set of parameters that fulfill this condition is clearly bounded by segments of the lines defined by $l_j(\alpha, \mu) = 1$. The shape of this set of parameters for which control is successful depends only on the local stability properties of the uncontrolled orbit; cf [18] for a more detailed discussion. Let $\text{Fix}_h^*(f, p)$ denote the set of SPFC-stabilizable orbits and since SPFC is a proper extension of PFC, we have $\text{Fix}_g^*(f, p) \subset \text{Fix}_h^*(f, p)$, i.e. any PFC-stabilizable periodic orbit is also SPFC-stabilizable.

4. Performance of stalled predictive feedback control

We illustrate the performance increase of SPFC by calculating the quantities defined above for the example of the Hénon map (3). From the stability function, we see that a stalling parameter α different from zero indeed leads to a drastic increase in optimal asymptotic convergence speed, cf figure 4. With $\varrho_{\min}^h(x^*) = \inf_{\mu, \alpha} \varrho_{x^*}(\alpha, \mu)$ denoting the smallest spectral radius, we calculated the mean

$$\rho_h(p) = 1 - \langle \varrho_{\min}^h(x^*) \rangle_{\text{Fix}_h^*(f, p)}, \quad (9)$$

minimum $\underline{\rho}_h(p)$ and maximum $\overline{\rho}_h(p)$ of $1 - \varrho_{\min}^h(x^*)$ for $x^* \in \text{Fix}_h^*(f, p)$ analogous to the quantities defined above to assess the scaling of optimal asymptotic convergence speed across different periods. The results are depicted in figure 5. In contrast to the exponential decrease in best asymptotic convergence speed for PFC, we obtain a period-independent scaling when stalling PFC even for the worst convergence speed $\underline{\rho}_h(p)$.

At the same time, we calculated the fraction of stabilizable fixed points given by

$$v_h(p) = \frac{\#(\text{Fix}_h^*(f, p))}{\#(\text{Fix}(f, p))}, \quad v_g(p) = \frac{\#(\text{Fix}_g^*(f, p))}{\#(\text{Fix}(f, p))}$$

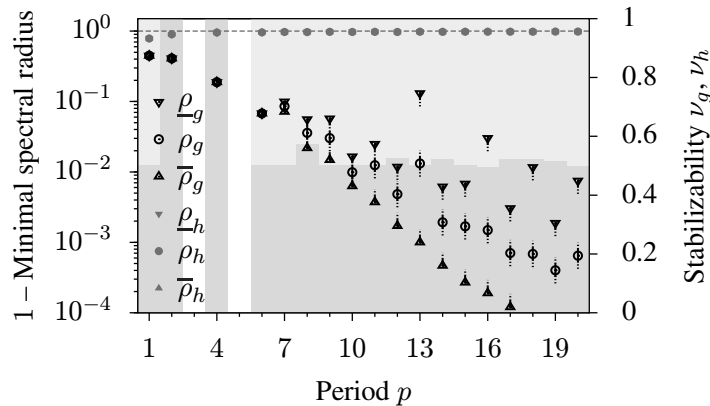


Figure 5. Stalling PFC leads to a period-invariant scaling of optimal asymptotic convergence speed. The shading indicates the fraction of PFC-stabilizable orbits (dark gray) and the fraction of periodic orbits that can only be stabilized by SPFC with a nonzero stalling parameter $\alpha > 0$ (light gray).

where $\#$ denotes the cardinality of a set. For the Hénon map, we see that the fraction of stabilizable orbits is increased from about one half for the original PFC to one for SPFC. In fact, it can be shown that for a two-dimensional system with periodic orbits of saddle type, all periodic orbits can be stabilized [18].

In order to relate these theoretical bounds to the actual performance achieved in an implementation, we performed simulations for the case without stalling $\alpha = 0$ (corresponding to the iteration of $g_{\mu,p}$), and for nonzero stalling parameters $\alpha = 3^{-1}$ ($m = 2, n = 1$ in (5)) and $\alpha = (1 + p)^{-1}$ ($m = p, n = 1$). To simulate real-world implementation where control would be turned on at a random point in time-initial conditions were distributed randomly on the attractor⁷. We calculated the smallest convergence time for the parameter values where reliability, i.e. the fraction of convergent runs, was at least 0.95. As shown in figure 6, SPFC yields lower convergence times for any period. Although the scaling of convergence times is not period-invariant as calculated for the linearized system, the increase is clearly smaller than for the original PFC. This improved scaling for SPFC allows for the successful stabilization of periodic orbits of most-large periods where PFC fails, in particular if the stalling parameter is chosen to be period dependent.

5. Discussion

With increasing dimension of the dynamical system the number of constraints on the local stability properties as given by (7) to stabilize periodic orbits also increases. While other two-dimensional hyperbolic chaotic maps essentially behave the same as the Hénon map due to their periodic orbits of saddle type, three- or higher-dimensional maps may exhibit different types of periodic orbits. More precisely, even with a nonzero stalling parameter not all the periodic orbits may be stabilized. However, the qualitative results regarding convergence speed remain the same

⁷ We iterated $h_{\mu,p}$ for 250 initial conditions distributed randomly on the attractor according to the dynamics of f . Convergence time t was determined by $\|x_t - x_{t+1}\| < 10^{-12}$ under the additional condition $\|f_p(x_t) - x_t\| < 10^{-6}$ up to a timeout of $t_{TO} = 4000$. All convergence times were rescaled to evaluations of f_p .

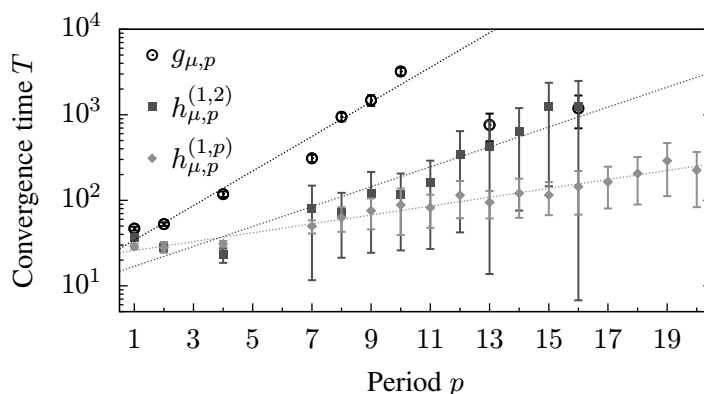


Figure 6. SPFC yields lower convergence times and an improved scaling across periods compared to PFC for both period-dependent and fixed stalling parameter. Mean and standard deviation are plotted together with approximate fits with an exponential function (dashed lines).

as indicated by a three-dimensional example with periodic orbits with two-dimensional unstable manifold [18]. Firstly, there is a large subset of stabilizable periodic orbits for which asymptotic convergence speed is essentially period independent as described above. Secondly, stalling PFC increases the number of stabilizable periodic orbits. *A priori* estimates of the local stability properties and calculation of attractor dimensions would be desirable for a determination of the efficiency of SPFC.

For a special choice of the stalling parameter, SPFC is related to what has been introduced as ‘rhythmic’ or ‘oscillating’ control, which were considered to increase the number of stabilizable periodic orbits [14] or to take into account time-delayed measurement of a system [15]. Oscillating feedback corresponds to a fixed value of $\alpha = 2^{-1}$ and has been investigated with respect to the number of stabilizable periodic orbits only. Although with this choice of parameter one can stabilize more periodic orbits in one or two dimensions (within a very small range of μ , cf figure 4), only the generalization to an arbitrary stalling parameter presented here yields fast stabilization of periodic orbits with higher-dimensional unstable manifolds [18]. SPFC is also related to ‘act-and-wait’ control [19] and, moreover, to ‘intermittent’ control [20] for linear control problems where the control signal is turned on and off periodically. Even though SPFC may be applicable in the same context, i.e. to time continuous systems through discretization, it primarily aims at stabilizing many unstable periodic orbits of a given nonlinear system using a one-parameter feedback control scheme.

How does one find a stalling and control parameter for (fast) convergence? Since increasing the stalling parameters m, n results in more evaluations of f to perform a single step of the iteration, one might want to choose α to keep them as small as possible. Moreover, adaptation mechanisms provide a way to tune the parameters of the system online to achieve fast convergence. In contrast to existing adaptation approaches for chaos control [5, 21] one would like to find the parameter values for which convergence is fastest. In fact, as we show in a forthcoming paper [18], gradient adaptation can be used not only to find the regime of the control parameter in which convergence can take place but also to optimize for speed.

An adaptation approach lifts the requirement to fine tune the parameter values *a priori*, making the method applicable to a wide variety of chaotic systems.

It would be interesting to see SPFC be applied to experimental systems, for example lasers. Such experimental setups are typically influenced by environmental noise and the control of noisy systems is an extensive topic [3]. Due to the lack of a straightforward criterion for successful convergence when noise is present, the treatment of noisy dynamics is beyond the scope of this paper. In the future, an analysis of SPFC for noisy systems may give important insights in the applicability of SPFC to experimental setups.

In conclusion, stalling PFC enhances its performance by both improving convergence speed and increasing the number of periodic orbits that can be stabilized while maintaining its ease of implementation. Hence, SPFC provides a non-invasive chaos control scheme that is broadly applicable since it requires little prior knowledge about the system and is capable of stabilizing many periodic orbits even of large periods.

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