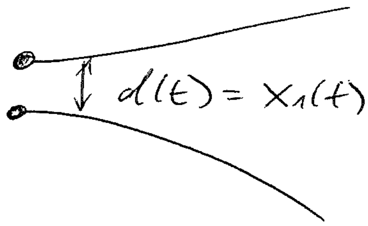
Chaos

1. sensitive dependence on initial conditions, yet dynamics stays in bounded domain.
   ↓
   formally: positive Ljapunov exponents.



$d(t) = x_1(t) - x_2(t) \sim d_0 \exp \lambda t$.

How possible? stretching in one direction, compression in the other

2. deterministic

3. aperiodic long-term behavior.
   (no fixed points, periodic orbits, quasiperiodic orbits).

$\equiv$ deterministic chaos.

- attractor sets are often fractals.
- example (Lorenz, 1963)

$$\dot{x} = \sigma(y-x) \qquad \sigma = 10$$
$$\dot{y} = rx - y - xz \qquad b = 8/3$$
$$\dot{z} = xy - bz \qquad r = 28.$$

deterministic chaos with regular statistics

$n = 1$ — only fixed points

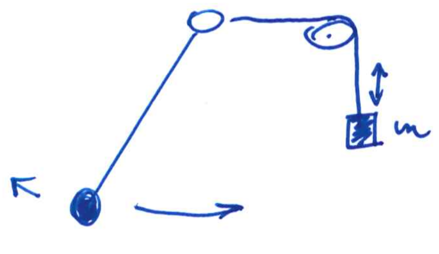
$n = 2$ — oscillations, no chaos

$n = 2$ — deterministic chaos

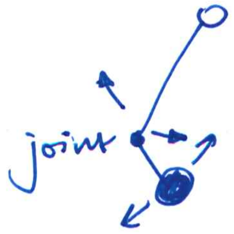
$n >> 10^3$ — statistical mechanics

chaos → nonlin. dyn. of mesoscopic degrees of freedom + epistemic noise

- Swinging Atwood machine



- Double pendulum

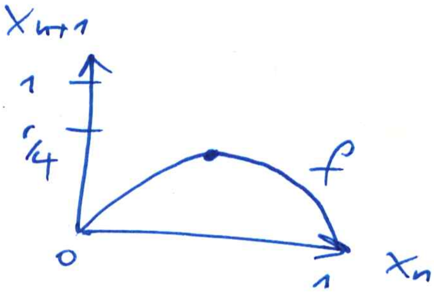


1D. maps. (discrete time)

$$x_{n+1} = f(x_n)$$

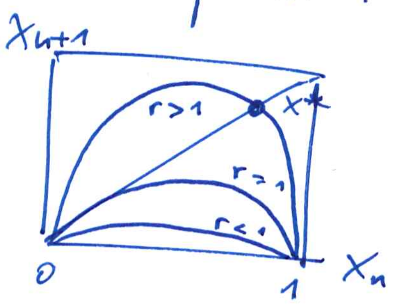
Logistic map.

$$x_{n+1} = r x_n (1 - x_n) = f(x_n)$$



Q: fixed points?

A:



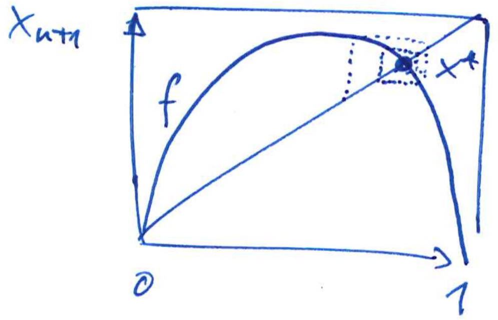
$x^* = 0$ for $r \leq 1$.

$x^* > 0$ for $r > 1$.

$f(x^*) = x^* \Rightarrow x^* = \frac{r-1}{r}$

Q: stability?

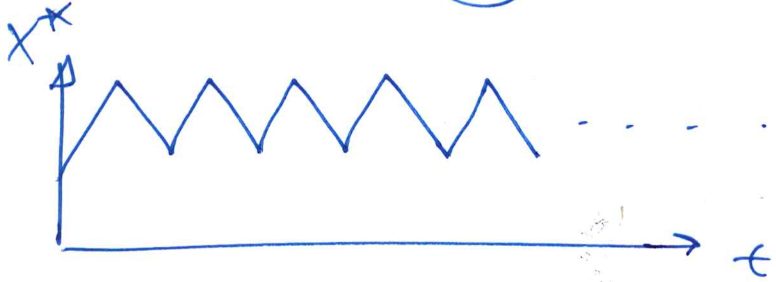
A: x^* stable $\Leftrightarrow f'(x^*) > -1 \Leftrightarrow r < 3$.

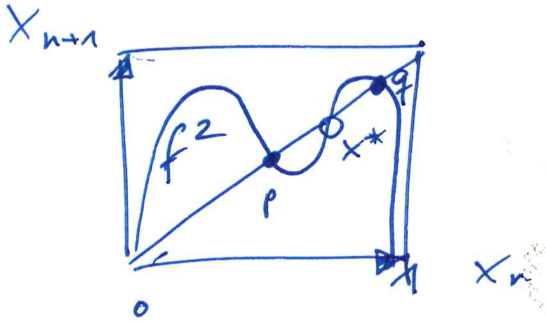


- start with $x_n = x^* - \varepsilon$
- cobweb construction
- $\ln |f'|$ plays role of Ljapunov exponent.

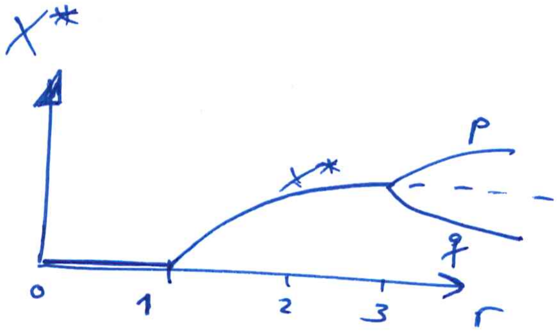
Q: What happens for $r > 3$?

A: period-doubling bifurcation





$\Rightarrow$



$$f^2(x) = x \iff$$

$$r^2 x(1-x)[1 - rx(1-x)] = x. \Rightarrow$$

factor out trivial solutions.

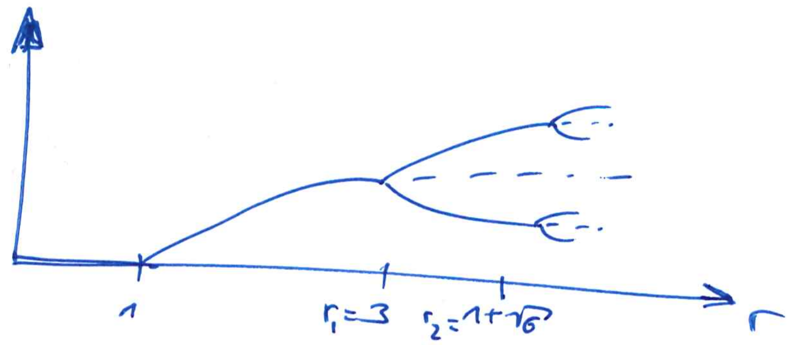
$x = 0$ and $x = \frac{r-1}{r}$

$$p, q = \frac{r+1 \pm \sqrt{(r-1)(r+1)}}{2r}$$

@ $r = 3$: $p = q = x^*$

Q: What happens for larger r?

A:



- period doubling bifurcation at r_n
- $\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669.$
- $r > r_\infty$: chaos (= aperiodic motion)

$\Rightarrow$ period-doubling route to chaos.

- even for $r > r_\infty$, there are window of periodic motion.

- periodic orbits appear in universal sequence

  $$1, 2, 2\times 2, 6, 5, 3, 2\times 3, \ldots$$

- for r slightly below lower bound of periodic window, one observes intermittency:

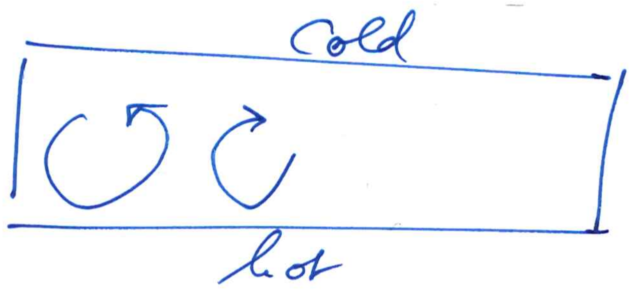
long periods of almost periodic motion — bursts of chaotic behavior — again periods of periodic motion.

- Metropolis: universal seq. for all unimodal maps $x_{n+1} = r f(x_n)$, $f(0)=f(1)=0$.

- Feigenbaum: $\delta = 4.669$ for doubling bifurcation for all unimodal maps.

Experimental test

- Convection rolls. (Libchaber)



- diodes, Josephson junctions, ...

Logistic map continued:
The period-doubling route to chaos or Renormalization theory for pedestrians

Logistic map:

$$x_{n+1} = r x_n (1-x_n), \quad r \ldots \text{ control parameter.}$$
$$= f(x_n)$$

$r = r_k$: birth of 2^k-cycle.

Let $\mu_k = r_k - r_1$.

Want to show: μ_k converge to μ_∞.
$\mu_k - \mu_\infty$ geom. series.

Task: Compute μ_k iteratively.

Start: We know $\mu_1 = 0$.

$\Rightarrow$ Normal form of period-doubling bifurcation.

$$y = x - x^*, \quad x^* = f(x^*)$$

$\Rightarrow$

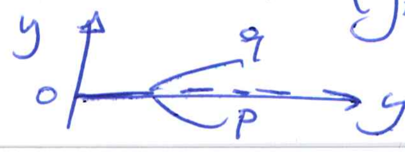
$$y_{n+1} = -(1+\mu) y_n + a y_n^2 + \ldots$$

$\Rightarrow$ wlog $a = 1$

$$\boxed{y_{n+1} = -(1+\mu)\mu_n + y_n^2 = g(y_n)}$$

$\Rightarrow$ Find fixed point of 2^{nd} iterate ($y_{n+2} = y_n$)

$$p, q = \frac{\mu \pm \sqrt{\mu^2 + 4\mu}}{2}$$



What happens if we add a perturbation to p?

$$p: \quad p = g^2(p).$$

$$p+\varepsilon_n: \quad p + \varepsilon_{n+1} = g^2(p+\varepsilon_n)$$
$$= p + (1-4\mu+\mu^2)\varepsilon_n + C\varepsilon_n^2$$
$$C = 4\mu + \mu^2 - 3\sqrt{\mu^2+4\mu}.$$

Set $\tilde{y}_n = C\varepsilon_n$

$\Rightarrow$
$$\tilde{y}_n = (1-4\mu-\mu^2)\tilde{y}_n + \tilde{y}_n^2$$
$$= -(1+\tilde{\mu})\tilde{y}_n + \tilde{y}_n^2$$
$$= \tilde{g}(\tilde{y}_n) \quad \text{with } \tilde{\mu} = \mu^2 + 4\mu - 2.$$

| We found normal form again, but with renormalized μ. |

$\Rightarrow$ find fixed point of 2nd iterate ($\tilde{y}_{n+2} = \tilde{y}_n$)

$$\Rightarrow \tilde{p}, \tilde{q} = \frac{\tilde{\mu} \pm \sqrt{\tilde{\mu}^2 + 4\tilde{\mu}}}{2}$$

Note: The following are equivalent events.

(i) birth of 2^k-cycle of g at μ_k

(ii) birth of 2^{k-1} cycle of g^2 at $\tilde{\mu}_{k-1}$

(iii) birth of 2^{k-2} cycle of g^4 at $\tilde{\tilde{\mu}}_{k-2}$

$\vdots$

We have

$$\tilde{\mu}_{k-1} = \mu_k^2 + 4\mu_k - 2 = h(\mu_k).$$

$$\tilde{\tilde{\mu}}_{k-\ell} = \underbrace{h \circ \dots \circ h}_{\ell}(\mu_k).$$

Now, we can work backwords.

$$0 = \tilde{\mu}_1 = h(\mu_2) \Rightarrow \mu_2 = -2 + \sqrt{6}.$$

$$0 = \tilde{\tilde{\mu}}_1 = h(\tilde{\mu}_2) = h(h(\mu_3)) \Rightarrow \mu_3 = h^{-1}(\mu_2)$$

$\vdots$

$$\Rightarrow \boxed{\mu_{k-1} = h(\mu_k)}$$

Fixed point $\mu_\infty = h(\mu_\infty) \Rightarrow$

$\mu_\infty \approx 0.56.$ (true 0.57).

Next, $\mu_k - \mu_\infty$ decays as geom. series.

$$\delta = \lim_{k \to \infty} \frac{\mu_{k-1} - \mu_\infty}{\mu_k - \mu_\infty} \underline{\underline{\text{l'Hospital}}}$$

$$\left.\frac{d\mu_{k-1}}{d\mu_k}\right|_{\mu = \mu_\infty} = 2\mu_\infty + 4.$$

≈ 5.12 (true 4.67)

10% off.

Renormalization for pedestrians.

$\mu_k = r_k - r_1$: birth of 2^k-cycle.

Q: Can we compute μ_k?

A: We know $\mu_1 = 0$.

Let $y = x - x^*$:

$$y_{n+1} = -(1+\mu) y_n + a y_n^2 + \dots$$

Wlog. $a = 1$.

$$(*) \quad y_{n+1} = -(1+\mu) y_n + y_n^2,$$

$$p, q = \frac{\mu \pm \sqrt{\mu^2 + 4\mu}}{2}$$

$$p + \eta_{n+1} = f^2(p + \eta_n)$$
$$= p + (1 - 4\mu - \mu^2)\eta_n + C \eta_n^2$$

$$C = 4\mu + \mu^2 - 3\sqrt{\mu^2 + 4\mu}.$$

Set

$$\tilde{y}_n = C \eta_n,$$

$$(**) \quad \tilde{y}_{n+1} = (1 - 4\mu - \mu^2)\tilde{y}_n + \tilde{y}_n^2$$
$$= -(1+\tilde{\mu})\tilde{y}_n + \tilde{y}_n \quad \text{with } \tilde{\mu} = \mu^2 + 4\mu - 2$$

birth of $2^2 = 4$ - cycle:

$$\tilde{\mu} = 0 \iff \mu_1 = \mu_2^2 + 4\mu_2 - 2$$

$$\Rightarrow \mu_2 = -2 + \sqrt{6}!$$

Similarly,

$$\mu_{k-1} = \mu_k^2 + 4\mu_k - 2.$$

fixed point $\mu_\infty \approx 0.56$ (true 0.57).

$$\delta = \lim_{k\to\infty} \frac{\mu_{k-1} - \mu^*}{\mu_k - \mu^*} \overset{\text{L'Hospital}}{=} \left.\frac{d\mu_{k-1}}{d\mu_k}\right|_{\mu=\mu^*}$$

$$= 2\mu^* + 4 \approx 5.12 \text{ (true } 4.67)$$

Introduction to fractals: The Cantor set.

- Consider the stoch. map.

$$x \in [0,1] \longmapsto \cancel{f(x)}$$

50% ↙ ↘ 50%

$f(x) = x/3$ $\quad$ $f(x) = 2/3 + x/3$.

Q: Is there an attractor set?

A: After N iterations

$$f^N(x) = 0,\underbrace{2022020\ldots 02}_{\substack{N\text{ digits}\\ \text{only } 0 \,\&\, 2}}\ \underbrace{102\ldots}_{\substack{\text{original}\\ \text{digits of } x.\\ \text{shifted to}\\ \text{the right.}}}$$

Symbolic dynamics

in base 3.

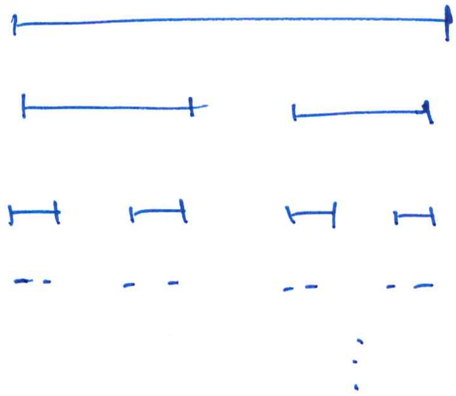
Each random seq. of finite length will appear with finite probability

$\Rightarrow$ Each $x_\infty \in [0,1]$ that can be written with only digits 0 & 2 in base 3 will be approximated by some $f^N(x)$ to arbitrary precision

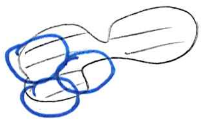
$\Rightarrow$ attractor set = Cantor set.

- Geometric construction of Cantor set

Step 0
Step 1
Step 2
Step 3



- Fractal dimension of a set $B \subseteq \mathbb{R}^n$
  - cover B with small disks/spheres
  - of radii $r_i < \delta$.



  - sum α-dimensional volumes

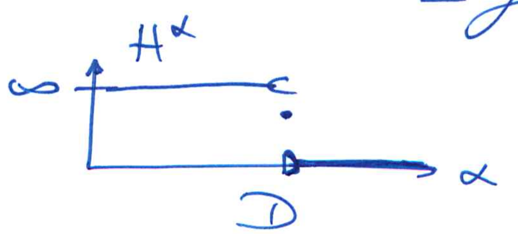
$$\omega_\alpha \sum_i r_i^\alpha \quad , \quad \omega_\alpha = \frac{\pi^{\alpha/2}}{\Gamma(1+\alpha/2)}$$

$$\omega_0 = 1, \ \omega_1 = 2, \ \omega_2 = \pi, \ \omega_3 = \tfrac{4}{3}\pi, \ \dots$$

  - take minimum over all possible covers. (and let $\delta \to 0$):

$$H^\alpha(B) = \lim_{\delta \searrow 0} \inf_{\{r_i\}} \omega_\alpha \sum_i r_i^\alpha$$

$\equiv$ generalization of Lebesgue meas.



D = Haussdorff-Besicovich-dim..

Q: What is the fractal dimension of the Cantor set?

$$C = C_{left} \cup C_{right}$$

$$H^\alpha(C) = H^\alpha(C_{left}) + H^\alpha(C_{right}).$$
$$= 2\, H^\alpha(C_{left})$$

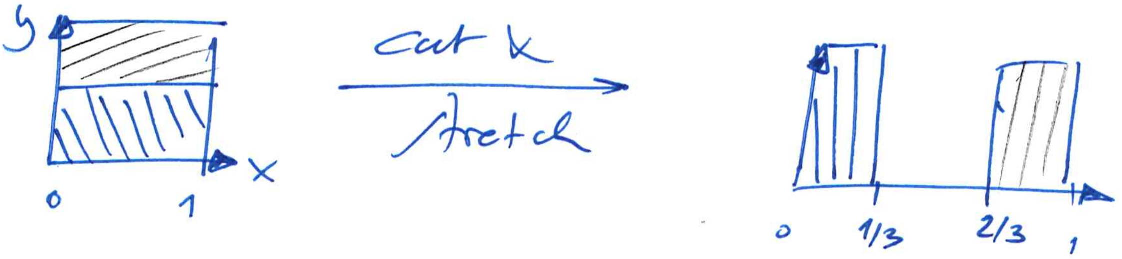
but also

$$H^\alpha(C_{left}) = \left(\tfrac{1}{3}\right)^\alpha H^\alpha(C)$$

$\Rightarrow$ $2 = 3^D$ if $0 < H^D(C) < \infty$

$\Rightarrow$ $D = \dfrac{\ln 2}{\ln 3} < 1.$

- Deterministic realisation: Baker's map.



Fractal dimension of exp. data.

- cover with cubes of size ε.

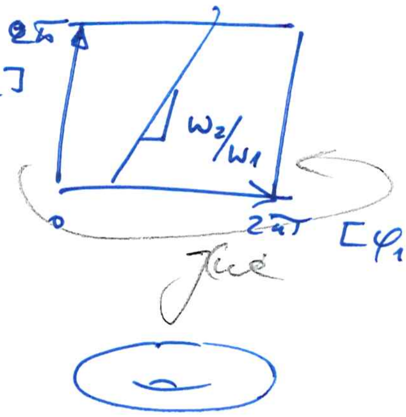
$$D_0 = \lim_{\varepsilon \downarrow 0} \frac{N(\varepsilon)}{\ln(1/\varepsilon)}$$

- Often $D_0 = D_{Hausdorff}$.

Higher order synchronization and Arnold tongues.

example
$$\dot{\varphi}_1 = \omega_1, \qquad \dot{\varphi}_2 = \omega_2$$
dynamics on torus

no coupling yet



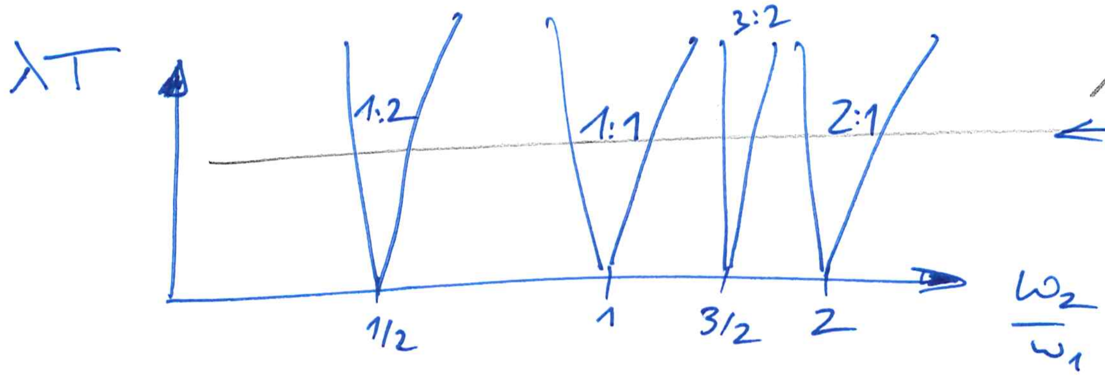
- rational case: $\frac{\omega_1}{\omega_2} = \frac{n}{m} \in \mathbb{Q} \Rightarrow$ higher order phase-locking $\dot{\varphi}_1 : \dot{\varphi}_2 = n : m$

- irrational case: quasi periodicity $\equiv$ finite number of incommensurable freq.

Q: What happens for oscillator coupling

$$\dot{\varphi}_i = \omega_i - \frac{\lambda}{2} \sin(\varphi_i - \varphi_j)$$

$$\dot{\delta} = \omega_1 - \omega_2 - \lambda \sin \delta, \quad \delta = \varphi_1 - \varphi_2 .$$



$\equiv \forall x_0 \forall \varepsilon$
$|C \cap B_\varepsilon(x_0)| > 0$
$\wedge\ |C \setminus B_\varepsilon(x_0)| > 0.$

← 'fat fractal'

← phase-locking in region of Lebesgue measure 0.

This was proven mathematically for the discrete-time Adler eqn.

$$\delta_{n+1} = \delta_n + \underbrace{\Delta\omega T}_{w} - \underbrace{\lambda T}_{k} \sin \delta_n \mod 2\pi$$

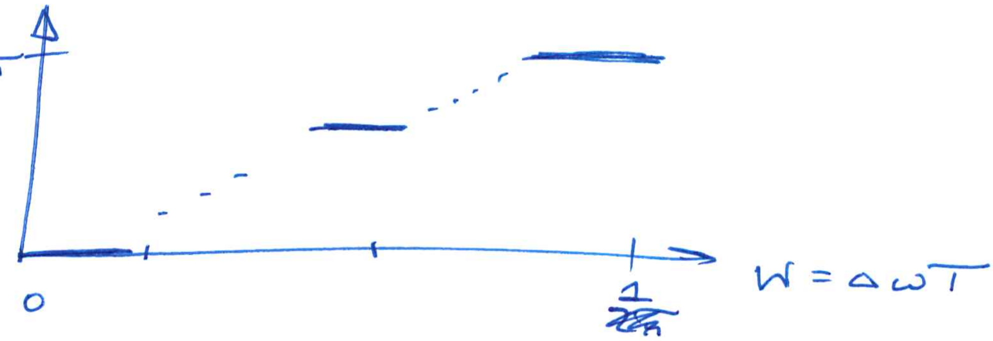
$\equiv$ circle map (Arnold 1965).

- Whether phase-locking occurs for given irrational w depends how well w can be approximated by rationals $\longrightarrow$ number theory, golden section $\frac{\sqrt{5}-1}{2}$ is worst.
- $k = -\lambda T = 1$: phase-locking region has Lebesgue measure 2π yet some quasi-periodic orbits remain.

Devil's staircase

rotation number

$$\lim_{m\to\infty} \frac{\theta_m - \theta_0}{m}$$



≡ continuous function which takes infinitesimal many infinitesimal small jumps.